Second order perturbations of a macroscopic string: Covariant approach

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Using a world-sheet covariant formalism, we derive the equations of motion for second order perturbations of a generic macroscopic string, thus generalizing previous results for first order perturbations. We give the explicit results for the first and second order perturbations of a contracting near-circular string; these results are relevant for the understanding of the possible outcome when a cosmic string contracts under its own tension, as discussed in a series of papers by Vilenkin and Garriga. In particular, second order perturbations are necessary for a consistent computation of the energy. We also quantize the perturbations and derive the mass formula up to second order in perturbations for an observer using world-sheet time τ. The high frequency modes give the standard Minkowski result while, interestingly enough, the Hamiltonian turns out to be non-diagonal in oscillators for low-frequency modes. Using an alternative definition of the vacuum, it is possible to diagonalize the Hamiltonian, and the standard string mass spectrum appears for all frequencies. We finally discuss how our results are also relevant for the problems concerning string-spreading near a black hole horizon, as originally discussed by Susskind.

I. INTRODUCTION

It is well known that the classical string equations of motion in flat Minkowski space can be solved exactly using a conformal gauge (see for instance [1]). Moreover, the gauge constraints arising in the conformal gauge can be solved exactly by supplementing the conformal gauge with a light-cone gauge. However, for many purposes, especially in connection with macroscopic cosmic strings, the formalism of the conformal and light cone gauges, although mathematically tractable, is not particularly useful. First of all, the world-sheet time τ is generally not related to the preferred coordinate time t in a simple way. Second, although the longitudinal oscillations are expressed in terms of the transverse oscillations, and therefore do not represent independent physical degrees of freedom, the longitudinal oscillations are actually still present. Finally, for macroscopic cosmic strings there is often a natural separation of the degrees of freedom into ''slow modes'' and ''fast modes,'' and this separation is often more transparent in alternative gauges. In a curved spacetime the situation is even worse: The classical string equations of motion cannot generally be solved in the conformal gauge (nor in any other gauge) and it is generally not even possible to supplement the conformal gauge with a light cone gauge, since it would be inconsistent with the equations of motion.

For macroscopic strings with small oscillations it is usually much more convenient to use a formalism where the small oscillations are considered as perturbations. Instead of the conformal gauge one can make a more physical gauge choice where, from the beginning, only transverse oscilla-

tions are present. Moreover, the world-sheet time τ of the unperturbed macroscopic string can be directly identified with the preferred coordinate time t. Another advantage is that spacetime and world-sheet covariance can be maintained at all stages. The price to pay for such a nonconformal gauge choice is that the string equations of motion are nonlinear. However, that is not really a problem in a perturbative scheme where the equations of motion are to be solved order by order in the expansion around the zeroth order unperturbed macroscopic string.

The world-sheet covariant perturbative approach was developed in [2] for membranes and strings in flat Minkowski space and in de Sitter space (a noncovariant approach was previously developed in [3]). The results were generalized to arbitrary curved backgrounds in [4–6] (see also [7] for some recent developments). However, until now only first order perturbations around the zeroth order macroscopic string have been considered. This is perfectly enough for many purposes, but in certain cases it is necessary to consider also the second order perturbations. For instance, considering small perturbations around a contracting circular string, it is easy to see that there is no contribution to the total conserved energy to first order; the first order contribution simply integrates out. The first nonzero contribution (besides the zeroth order contribution) to the total energy is quadratic in the first order perturbations, but then also second order perturbations must be included for consistency, since they contribute to the same order.

The purpose of the present paper is first of all to generalize the results of [2,4–6] for first order perturbations to second order perturbations. That is, we derive the equations of motion for the second order perturbations in world-sheet covariant form. We then give the explicit results for the first and second order perturbations of a contracting near-circular string; these results are relevant for the understanding of the

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possible outcome when a cosmic string contracts under its own tension [2]. Moreover, as already mentioned, the second order perturbations are necessary also for a consistent computation of the total conserved energy. After obtaining explicitly the expression for the classical mass energy, we quantize the perturbations and derive the quantum mass formula up to second order in perturbations for an observer using world-sheet time τ. The high frequency modes give rise to the standard Mikowski result while, interestingly enough, the Hamiltonian turns out to be nondiagonal in oscillators for low-frequency modes. We then show that using an alternative definition of the vacuum, it is possible to diagonalize the Hamiltonian, and the standard string mass spectrum appears for all frequencies. We finally discuss how our results are also relevant for the problems concerning string-spreading near a black hole horizon, as originally discussed by Susskind [8].

II. GENERAL FORMALISM

Our starting point is the Nambu-Goto action

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-G}, \quad G \equiv \det(G_{AB}), \quad (2.1)$$

where G_{AB} is the induced metric on the string world sheet

$$G_{AB} \equiv \eta_{\mu\nu} x^\mu_{,A} x^\nu_{,B}. \quad (2.2)$$

Here $(A,B)=(0,1)$ are the world-sheet indices, while $(\mu,\nu)=(0,1,2,3)$ are the spacetime indices. We consider strings in 4-dimensional Minkowski space using Cartesian coordinates and sign conventions $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$.

The conjugate momentum P^A_μ is given by

$$P^A_\mu \equiv \frac{\delta\mathcal{L}}{\delta x^\mu_{,A}} = \frac{1}{2\pi\alpha'} \sqrt{-G} G^{AB} x_{\mu,B}. \quad (2.3)$$

The equations of motion, corresponding to the action (2.1), are then

$$\partial_A P^A_\mu = 0. \quad (2.4)$$

As is well known, the tangential projection of Eq. (2.4), $x^\mu_{,B} \partial_A P^A_\mu = 0$, is an identity; thus the equation of motion is equivalently

$$n^\mu_i \partial_A P^A_\mu = 0, \quad (2.5)$$

where the two normal vectors n^μ_i ($i=1,2$) are introduced by

$$\eta_{\mu\nu} n^\mu_i n^\nu_j = \delta_{ij}, \quad \eta_{\mu\nu} n^\mu_i x^\nu_{,A} = 0. \quad (2.6)$$

We shall be interested in physical situations where the solution to Eq. (2.4) [or, equivalently, Eq. (2.5)] is naturally described as a macroscopic string experiencing small perturbations. Up to second order perturbations, we therefore write x^μ in the following way:

$$x^\mu = \bar{x}^\mu + \delta x^\mu_{(1)} + \delta x^\mu_{(2)}. \quad (2.7)$$

Here, and in the following, the bar represents unperturbed (zeroth order) quantities. Moreover, since we are interested only in physical (transverse) perturbations, δx^μ can be expanded on the normal vectors, $\delta x^\mu = n^\mu_i \Phi^i$. Thus, by expanding up to second order

$$\delta x^\mu_{(1)} = \bar{n}^\mu_i \Phi^i_{(1)}, \quad (2.8)$$

$$\delta x^\mu_{(2)} = \bar{n}^\mu_i \Phi^i_{(2)} + \delta n^\mu_{i(1)} \Phi^i_{(1)}. \quad (2.9)$$

In Eq. (2.9) we need the first order perturbation of the normal vector, $\delta n^\mu_{i(1)}$. It is easily obtained from Eq. (2.6)

$$\delta n^\mu_{i(1)} = -(\bar{D}^A_{ij} \Phi^j_{(1)}) \bar{x}^\mu_{,A}, \quad (2.10)$$

where the covariant derivative $\bar{D}_{ijA}$ is defined by

$$\bar{D}_{ijA} \equiv \delta_{ij} \bar{\nabla}_A + \bar{\mu}_{ijA}. \quad (2.11)$$

Here $\bar{\nabla}_A$ is the covariant derivative with respect to the induced metric $\bar{G}_{AB}$ on the unperturbed world sheet, while $\bar{\mu}_{ijA}$ is the torsion (normal fundamental form) of the unperturbed world sheet

$$\bar{\mu}_{ijA} \equiv \eta_{\mu\nu} \bar{n}^\mu_i \bar{n}^\nu_{j,A}. \quad (2.12)$$

The equations of motion of the first and second order perturbations are conveniently written in terms of geometric quantities such as the covariant derivative $\bar{D}_{ijA}$ and the extrinsic curvature (second fundamental form) $\bar{\Omega}_{iAB}$ of the unperturbed world sheet

$$\bar{\Omega}_{iAB} \equiv \eta_{\mu\nu} \bar{n}^\mu_i \bar{x}^\nu_{,AB}. \quad (2.13)$$

This makes manifest the world-sheet covariance as well as the SO(2) invariance under rotations of the normal vectors. Some useful formulas are given in the Appendix. Here we just list the results for the conjugate momentum P^A_μ.

To zeroth order, it is simply

$$\bar{P}^A_\mu = \frac{1}{2\pi\alpha'} \sqrt{-\bar{G}} \bar{G}^{AB} \bar{x}_{\mu,B}. \quad (2.14)$$

The first order perturbation is also easily obtained

$$\delta P^A_{\mu(1)} = \frac{1}{2\pi\alpha'} \sqrt{-\bar{G}} ((\bar{D}^A_{ij} \Phi^j_{(1)}) \bar{n}^i_\mu$$
$$+ (\bar{\Omega}^{AB}_i - \bar{G}^{AB} \bar{\Omega}^C_{iC}) \Phi^i_{(1)} \bar{x}_{\mu,B}). \quad (2.15)$$

The second order perturbation is considerably more complicated. Using the formulas of the Appendix, it becomes

$$\delta P^A_{\mu(2)} = \frac{1}{2\pi\alpha'}\sqrt{-\bar{G}}\Bigg((\bar{D}^A_{ij}\Phi^j_{(2)})\bar{n}^i_\mu + (\bar{\Omega}^{AB}_i - \bar{G}^{AB}\bar{\Omega}^C_{iC})\Phi^i_{(2)}\bar{x}_{\mu,B} + (\bar{D}^{jB}_k\bar{D}^A_{ji}\Phi^i_{(1)})\Phi^k_{(1)}\bar{x}_{\mu,B} - \bar{G}^{AB}(\bar{D}^{jC}_k\bar{D}_{jiC}\Phi^i_{(1)})\Phi^k_{(1)}\bar{x}_{\mu,B}$$

$$-\frac{1}{2}\bar{G}^{AB}(\bar{D}^C_{kj}\Phi^j_{(1)})(\bar{D}^k_{iC}\Phi^i_{(1)})\bar{x}_{\mu,B} - (\bar{D}^A_{ik}\Phi^k_{(1)})\bar{\Omega}^C_{jC}\Phi^j_{(1)}\bar{n}^i_\mu + 2(\bar{D}_{ikB}\Phi^k_{(1)})\bar{\Omega}^{AB}_j\Phi^j_{(1)}\bar{n}^i_\mu - (\bar{D}_{jkB}\Phi^k_{(1)})\bar{\Omega}^{AB}_i\Phi^j_{(1)}\bar{n}^i_\mu$$

$$+\frac{1}{2}\bar{G}^{AB}(\bar{\Omega}^C_{jC}\bar{\Omega}^D_{iD} - \bar{\Omega}_{jCD}\bar{\Omega}^{CD}_i)\Phi^j_{(1)}\Phi^i_{(1)}\bar{x}_{\mu,B} + (\bar{\Omega}^{AC}_j\bar{\Omega}^B_{iC} - \bar{\Omega}^C_{jC}\bar{\Omega}^{AB}_i)\Phi^j_{(1)}\Phi^i_{(1)}\bar{x}_{\mu,B}\Bigg). \tag{2.16}$$

Now it is straightforward to obtain the equations of motion (2.4) order by order in the expansion. To zeroth order, it is simply

$$\bar{\Omega}^C_{iC} = 0, \tag{2.17}$$

i.e., the well-known result of vanishing mean extrinsic curvature for a minimal surface.

To first order, the result is (using also the zeroth order equation of motion)

$$(\bar{D}^A_{jk}\bar{D}^k_{iA} + \bar{\Omega}_{jAB}\bar{\Omega}^{AB}_i)\Phi^i_{(1)} = 0 \tag{2.18}$$

as was already obtained independently in a number of papers [4,5].

The second order equation of motion, which to our knowledge has not been obtained before in covariant form, becomes (using also the zeroth and first order equations of motion)

$$(\bar{D}^A_{jk}\bar{D}^k_{iA} + \bar{\Omega}_{jAB}\bar{\Omega}^{AB}_i)\Phi^i_{(2)} = f_j, \tag{2.19}$$

where the source f_j is given in terms of the first order perturbations

$$f_j = -2(\bar{D}_{jiB}\bar{D}^i_{kA}\Phi^k_{(1)})\bar{\Omega}^{AB}_l\Phi^l_{(1)} - 2(\bar{D}_{jkA}\Phi^k_{(1)})$$
$$\times(\bar{D}^i_{lB}\Phi^l_{(1)})\bar{\Omega}^{AB}_i + (\bar{D}^i_{kA}\Phi^k_{(1)})(\bar{D}_{ilB}\Phi^l_{(1)})\bar{\Omega}^{AB}_j. \tag{2.20}$$

It should be mentioned that the derivation of the source term from Eqs. (2.4), (2.16) is a somewhat lengthy exercise in differential geometry. To finally obtain the source term in the relatively simple form, Eq. (2.20), we used among other things the completeness relation

$$\eta^{\mu\nu} = \bar{G}^{AB}\bar{x}^\mu_{,A}\bar{x}^\nu_{,B} + \delta^{ij}\bar{n}^\mu_i\bar{n}^\nu_j \tag{2.21}$$

the Weingarten equation

$$\bar{\nabla}_A\bar{\nabla}_B\bar{x}^\mu = \bar{\Omega}^i_{AB}\bar{n}^\mu_i \tag{2.22}$$

the Gauss-Codazzi equation

$$\bar{D}_{ijA}\bar{\Omega}^{jA}_B = \bar{D}_{ijB}\bar{\Omega}^{jA}_A \tag{2.23}$$

as well as the identity

$$2\bar{\Omega}^B_{iA}\bar{\Omega}^C_{jB}\bar{\Omega}^A_{kC} = \bar{\Omega}^B_{iA}\bar{\Omega}^A_{jB}\bar{\Omega}^C_{kC} + \bar{\Omega}^B_{jA}\bar{\Omega}^A_{kB}\bar{\Omega}^C_{iC} + \bar{\Omega}^B_{kA}\bar{\Omega}^A_{iB}\bar{\Omega}^C_{jC}$$
$$-\bar{\Omega}^A_{iA}\bar{\Omega}^B_{jB}\bar{\Omega}^C_{kC}. \tag{2.24}$$

Notice in particular that the zeroth order equation of motion ensures vanishing of the left hand sides of Eqs. (2.23),(2.24).

III. CIRCULAR STRING

We now consider the case where the unperturbed string is a circular string in the x-y plane. Defining R_0 to be the maximal radius, it is thus parametrized by

$$\bar{t} = R_0\tau,$$
$$\bar{x} = R_0\cos\tau\cos\sigma,$$
$$\bar{y} = R_0\cos\tau\sin\sigma, \tag{3.1}$$
$$\bar{z} = 0,$$

such that

$$\bar{G}_{AB} = \mathrm{diag}(-R_0^2\cos^2\tau, R_0^2\cos^2\tau). \tag{3.2}$$

The unperturbed normal vectors are given by

$$\bar{n}^\mu_1 = \begin{pmatrix} \sin\tau/\cos\tau \\ -\cos\sigma/\cos\tau \\ -\sin\sigma/\cos\tau \\ 0 \end{pmatrix}, \quad \bar{n}^\mu_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.3}$$

The only nonvanishing components of the unperturbed extrinsic curvature $\bar{\Omega}_{iAB}$ are

$$\bar{\Omega}_{1\tau\tau} = \bar{\Omega}_{1\sigma\sigma} = R_0, \tag{3.4}$$

while all components of the unperturbed torsion $\bar{\mu}_{ijA}$ vanish.

From Eqs. (3.2)–(3.4) follows that the zeroth order equation of motion, Eq. (2.17), is trivially satisfied.

The first order equation of motion, Eq. (2.18), reduces to

$$\left(-\partial^2_\tau + \partial^2_\sigma + \frac{2}{\cos^2\tau}\right)\Phi^1_{(1)} = 0, \tag{3.5}$$

$$(-\partial_\tau^2 + \partial_\sigma^2)\Phi_{(1)}^2 = 0. \qquad (3.6)$$

These equations are easily solved using Fourier expansions

$$\Phi_{(1)}^i(\tau,\sigma) = \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} C_n^i(\tau)e^{-in\sigma}, \quad C_{-n}^i = C_n^{i*}. \qquad (3.7)$$

The Fourier coefficients C_n^1 are given by

$$C_0^1 = \sqrt{2}(a_0^1(1 + \tau \tan \tau) + b_0^1 \tan \tau),$$

$$C_1^1 = a_1^1 \left(\frac{1}{\cos \tau} - \frac{i}{2} \left(\frac{\tau}{\cos \tau} + \sin \tau \right) \right)$$

$$+ b_1^{1*} \left(\frac{1}{\cos \tau} + \frac{i}{2} \left(\frac{\tau}{\cos \tau} + \sin \tau \right) \right),$$

$$C_n^1 = a_n^1 \frac{n + i \tan \tau}{\sqrt{n(n^2-1)}} e^{-in\tau} + b_n^{1*} \frac{n - i \tan \tau}{\sqrt{n(n^2-1)}} e^{in\tau}, \quad n \geq 2. \qquad (3.8)$$

The Fourier coefficients C_n^2 are given by

$$C_0^2 = \sqrt{2}(a_0^2 + b_0^2 \tau),$$

$$C_1^2 = a_1^2 e^{-i\tau} + b_1^{2*} e^{i\tau},$$

$$C_n^2 = \frac{1}{\sqrt{n}} a_n^2 e^{-in\tau} + \frac{1}{\sqrt{n}} b_n^{2*} e^{in\tau}, \quad n \geq 2. \qquad (3.9)$$

Recall that the negative n modes are defined in terms of the above ones by $C_{-n}^i = C_n^{i*}$. Notice also that the $n = 0, \pm 1$ modes are treated separately. This will be explained later; see also [2]. Moreover, the normalizations and precise definitions of the modes a_n^i, b_m^j in Eqs. (3.8),(3.9) are motivated as follows: The first order equations of motion, Eqs. (3.5),(3.6), correspond to the effective action

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \left\{ \Phi_{(1)}^1 \left(-\partial_\tau^2 + \partial_\sigma^2 + \frac{2}{\cos^2 \tau} \right) \Phi_{(1)}^1 \right.$$

$$\left. + \Phi_{(1)}^2 (-\partial_\tau^2 + \partial_\sigma^2) \Phi_{(1)}^2 \right\}. \qquad (3.10)$$

The conjugate momenta are defined by $\Pi_{(1)}^i \equiv \delta S / \delta \dot{\Phi}_{(1)}^i = \dot{\Phi}_{(1)}^i / 2\pi\alpha'$. They correspond to normal projections of the conjugate momenta $\delta P_{\mu(1)}^\tau$ introduced in Eq. (2.15). Then the canonical Poisson brackets are

$$\{\Pi_{(1)}^i(\tau,\sigma), \Phi_{(1)}^j(\tau,\sigma')\} = -\delta^{ij}\delta(\sigma - \sigma') \qquad (3.11)$$

and it is straightforward to show that the modes a_n^i, b_n^i obey

$$\{b_0^i, a_0^j\} = -\delta^{ij}, \qquad (3.12)$$

$$\{a_n^i, a_m^{j*}\} = \{b_n^i, b_m^{j*}\} = -i\delta^{ij}\delta_{nm}, \quad n, m \geq 1,$$

i.e., a_0^i, b_0^j are conventionally normalized center of mass coordinate and momenta, while a_n^i, b_m^j $(n, m \geq 1)$ are conventionally normalized harmonic oscillator modes.

We now come to the second order equation of motion, Eq. (2.19). In the present case, it reduces to

$$\left(-\partial_\tau^2 + \partial_\sigma^2 + \frac{2}{\cos^2 \tau} \right) \Phi_{(2)}^1 = R_0^2 \cos^2 \tau f^1, \qquad (3.13)$$

$$(-\partial_\tau^2 + \partial_\sigma^2)\Phi_{(2)}^2 = R_0^2 \cos^2 \tau f^2, \qquad (3.14)$$

where the source terms f^i, Eq. (2.20), are given by

$$R_0^2 \cos^2 \tau f^1 = \frac{-2}{R_0 \cos^2 \tau} \left(\Phi_{(1)}^1 \ddot{\Phi}_{(1)}^1 + \Phi_{(1)}^1 \Phi''^1_{(1)} \right.$$

$$\left. + \frac{2 \sin \tau}{\cos \tau} \Phi_{(1)}^1 \dot{\Phi}_{(1)}^1 \right)$$

$$- \frac{1}{R_0 \cos^2 \tau} (\dot{\Phi}_{(1)}^1 \dot{\Phi}_{(1)}^1 - \dot{\Phi}_{(1)}^2 \dot{\Phi}_{(1)}^2 + \Phi'^1_{(1)} \Phi'^1_{(1)}$$

$$- \Phi'^2_{(1)} \Phi'^2_{(1)}), \qquad (3.15)$$

$$R_0^2 \cos^2 \tau f^2 = \frac{-2}{R_0 \cos^2 \tau} \left(\Phi_{(1)}^1 \ddot{\Phi}_{(1)}^2 + \Phi_{(1)}^1 \Phi''^2_{(1)} \right.$$

$$\left. + \frac{2 \sin \tau}{\cos \tau} \Phi_{(1)}^1 \dot{\Phi}_{(1)}^2 \right) - \frac{2}{R_0 \cos^2 \tau} (\dot{\Phi}_{(1)}^1 \dot{\Phi}_{(1)}^2$$

$$+ \Phi'^1_{(1)} \Phi'^2_{(1)}). \qquad (3.16)$$

Equations (3.13),(3.14) are solved using Fourier expansions

$$\Phi_{(2)}^i(\tau,\sigma) = \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} D_n^i(\tau)e^{-in\sigma}, \quad D_{-n}^i = D_n^{i*}, \qquad (3.17)$$

$$-R_0^2 \cos^2 \tau f^i(\tau,\sigma)$$

$$= \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} f_n^i(\tau)e^{-in\sigma}, \quad f_{-n}^i = f_n^{i*} \qquad (3.18)$$

(the factor $-R_0^2 \cos^2 \tau$ is included in f^i for convenience). Then Eqs. (3.13),(3.14) become

$$\ddot{D}_n^1 + \left(n^2 - \frac{2}{\cos^2 \tau} \right) D_n^1 = f_n^1, \qquad (3.19)$$

$$\ddot{D}_n^2 + n^2 D_n^2 = f_n^2, \qquad (3.20)$$

where the source terms f_n^i are given by

$$f_n^1 = \frac{\sqrt{2\alpha'}}{R_0 \cos^2\tau} \sum_{m=-\infty}^{\infty} \left\{ C_m^1 \left(\ddot{C}_{n-m}^1 - (n-m)\left(n-\frac{m}{2}\right)C_{n-m}^1 + \frac{2\sin\tau}{\cos\tau}\dot{C}_{n-m}^1 \right) \right.$$
$$\left. + \frac{1}{2}(\dot{C}_m^1 \dot{C}_{n-m}^1 - \dot{C}_m^2 \dot{C}_{n-m}^2) + \frac{m(n-m)}{2}C_m^2 C_{n-m}^2 \right\}, \tag{3.21}$$

$$f_n^2 = \frac{\sqrt{2\alpha'}}{R_0 \cos^2\tau} \sum_{m=-\infty}^{\infty} \{ C_m^1 (\ddot{C}_{n-m}^2 - n(n-m)C_{n-m}^2 + 2\sin\tau\cos\tau\,\dot{C}_{n-m}^2) + \dot{C}_m^1 \dot{C}_{n-m}^2 \}. \tag{3.22}$$

By solving Eq. (3.19), we find that the Fourier coefficients D_n^1 are given by

$$D_0^1 = \tan\tau \int_0^\tau (1 + \tau' \tan\tau')f_0^1 d\tau' - (1 + \tau\tan\tau)\int_0^\tau \tan\tau' f_0^1 d\tau' + \sqrt{2}(\alpha_0^1(1 + \tau\tan\tau) + \beta_0^1\tan\tau),$$

$$D_1^1 = \frac{1}{2i}\left(\frac{1}{\cos\tau} + \frac{i}{2}\left(\frac{\tau}{\cos\tau} + \sin\tau\right)\right)\int_0^\tau \left(\frac{1}{\cos\tau'} - \frac{i}{2}\left(\frac{\tau'}{\cos\tau'} + \sin\tau'\right)\right)f_1^1 d\tau'$$
$$- \frac{1}{2i}\left(\frac{1}{\cos\tau} - \frac{i}{2}\left(\frac{\tau}{\cos\tau} + \sin\tau\right)\right)\int_0^\tau \left(\frac{1}{\cos\tau'} + \frac{i}{2}\left(\frac{\tau'}{\cos\tau'} + \sin\tau'\right)\right)f_1^1 d\tau'$$
$$+ \alpha_1^1\left(\frac{1}{\cos\tau} - \frac{i}{2}\left(\frac{\tau}{\cos\tau} + \sin\tau\right)\right) + \beta_1^{1*}\left(\frac{1}{\cos\tau} + \frac{i}{2}\left(\frac{\tau}{\cos\tau} + \sin\tau\right)\right),$$

$$D_n^1 = \frac{n - i\tan\tau}{2in(n^2-1)}e^{in\tau}\int_0^\tau (n + i\tan\tau')e^{-in\tau'}f_n^1 d\tau' - \frac{n + i\tan\tau}{2in(n^2-1)}e^{-in\tau}\int_0^\tau (n - i\tan\tau')e^{in\tau'}f_n^1 d\tau'$$
$$+ \alpha_n^1\frac{n + i\tan\tau}{\sqrt{n(n^2-1)}}e^{-in\tau} + \beta_n^{1*}\frac{n - i\tan\tau}{\sqrt{n(n^2-1)}}e^{in\tau}, \quad n \geq 2. \tag{3.23}$$

By solving Eq. (3.20), we find that the Fourier coefficients D_n^2 are given by

$$D_0^2 = \tau\int_0^\tau f_0^2 d\tau' - \int_0^\tau \tau' f_0^2 d\tau' + \sqrt{2}(\alpha_0^2 + \beta_0^2\tau),$$

$$D_1^2 = \frac{1}{2i}e^{i\tau}\int_0^\tau e^{-i\tau'}f_1^2 d\tau' - \frac{1}{2i}e^{-i\tau}\int_0^\tau e^{i\tau'}f_1^2 d\tau' + \alpha_1^2 e^{-i\tau} + \beta_1^{2*}e^{i\tau},$$

$$D_n^2 = \frac{1}{2in}e^{in\tau}\int_0^\tau e^{-in\tau'}f_n^2 d\tau' - \frac{1}{2in}e^{-in\tau}\int_0^\tau e^{in\tau'}f_n^2 d\tau' + \frac{1}{\sqrt{n}}\alpha_n^2 e^{-in\tau} + \frac{1}{\sqrt{n}}\beta_n^{2*}e^{in\tau}, \quad n \geq 2. \tag{3.24}$$

Again, we recall that the negative n modes are defined in terms of the above ones by $D_{-n}^i = D_n^{i*}$.

This completes the exact explicit solution of the first and second order perturbations around the circular string.

IV. MASS SPECTRUM

In this section we shall consider some physical quantities for the string, including energy $\mathcal{E}$, momentum $\vec{P}$ and angular momentum T_ν^μ. They are all given in terms of the conjugate momentum introduced in Eq. (2.3)

$$\mathcal{E} \equiv \int_0^{2\pi} d\sigma P_t^\tau, \tag{4.1}$$

$$\vec{P} \equiv \int_0^{2\pi} d\sigma \vec{P}^\tau, \tag{4.2}$$

$$T_\nu^\mu \equiv \int_0^{2\pi} d\sigma (x^\mu P_\nu^\tau - x_\nu P^{\tau\mu}) \tag{4.3}$$

and they are all conserved

$$\dot{\mathcal{E}}=0, \quad \dot{\vec{P}}=0, \quad \dot{T}^\mu_\nu=0 \tag{4.4}$$

as follows from Eq. (2.4).

We shall also obtain the mass spectrum from the classical mass formula $M^2=\mathcal{E}^2-\vec{P}^2$. Up to second order perturbations, it reads

$$M^2=(\bar{\mathcal{E}}+\delta\mathcal{E}_{(1)}+\delta\mathcal{E}_{(2)}+\cdots)^2-(\bar{\vec{P}}+\delta\vec{P}_{(1)}+\delta\vec{P}_{(2)}+\cdots)^2$$

$$=(\bar{\mathcal{E}}^2-\bar{\vec{P}}^2)+2(\bar{\mathcal{E}}\delta\mathcal{E}_{(1)}-\bar{\vec{P}}\,\delta\vec{P}_{(1)})$$

$$+(2\bar{\mathcal{E}}\delta\mathcal{E}_{(2)}+\delta\mathcal{E}_{(1)}\delta\mathcal{E}_{(1)}-2\bar{\vec{P}}\,\delta\vec{P}_{(2)}-\delta\vec{P}_{(1)}\delta\vec{P}_{(1)})$$

$$+\cdots. \tag{4.5}$$

The physical quantities, of course, will depend on the particular choice of initial conditions. The initial conditions are fixed in terms of the maximal radius R_0 (zeroth order), the modes a^i_n, b^j_m (first order) and the modes α^i_n, β^j_m (second order).

To zeroth order, we get from Eq. (2.3) and Eqs. (3.1),(3.2)

$$\bar{P}^\tau_\mu=\frac{-1}{2\pi\alpha'}\dot{\bar{x}}_\mu \tag{4.6}$$

such that

$$\bar{\mathcal{E}}=R_0/\alpha', \quad \bar{\vec{P}}=0, \quad \bar{T}^\mu_\nu=0, \tag{4.7}$$

i.e., only a zeroth order contribution to the energy.

To first order we get

$$\delta P^\tau_{\mu(1)}=\frac{-1}{2\pi\alpha'}\left(\delta_{ij}\dot{\Phi}^i_{(1)}\bar{n}^j_\mu-\frac{1}{R_0\cos^2\tau}\Phi^1_{(1)}\dot{\bar{x}}_\mu\right) \tag{4.8}$$

from which we can easily obtain the first order contributions to $\mathcal{E}$, $\vec{P}$, and T^μ_ν. However, following the discussion of [2], we take as initial conditions for the first order perturbations

$$a^i_0=b^i_0=a^i_{\pm1}=b^i_{\pm1}=0, \quad i=1,2. \tag{4.9}$$

This is motivated by the fact that the $n=0,\pm1$ modes correspond to rigid spacetime translations and rotations of the circular string [2]. That is to say, they do not really correspond to actual perturbations but merely to redefinitions of the initial circular string. Then, using Eq. (4.9), it is easy to show that

$$\delta\mathcal{E}_{(1)}=0, \quad \delta\vec{P}_{(1)}=0, \quad \delta T^\mu_{\nu(1)}=0, \tag{4.10}$$

i.e., no contributions at all to the physical quantities from the first order perturbations.

The second order contributions to the physical quantities can be straightforwardly computed, but here we shall give only the result for the mass squared. As follows from Eq. (4.5) and taking into account Eqs. (4.7), (4.10), it is necessary to compute only $\delta\mathcal{E}_{(2)}$. Using Eq. (2.16), we find that

$$\delta P^\tau_{t(2)}=\frac{1}{2\pi\alpha'}\left(\frac{\sin\tau}{\cos\tau}\dot{\Phi}^1_{(2)}-\frac{1}{\cos^2\tau}\Phi^1_{(2)}-\frac{2\sin\tau}{R_0\cos^3\tau}\Phi^1_{(1)}\dot{\Phi}^1_{(1)}\right.$$

$$+\frac{2}{R_0\cos^4\tau}\Phi^1_{(1)}\Phi^1_{(1)}$$

$$-\frac{1}{R_0\cos^2\tau}(\Phi^1_{(1)}\ddot{\Phi}^1_{(1)}+\Phi^2_{(1)}\ddot{\Phi}^2_{(1)})$$

$$+\frac{1}{2R_0\cos^2\tau}(\dot{\Phi}^1_{(1)}\dot{\Phi}^1_{(1)}+\dot{\Phi}^2_{(1)}\dot{\Phi}^2_{(1)}-{\Phi'}^1_{(1)}\Phi'^1_{(1)}$$

$$\left.-\Phi'^2_{(1)}\Phi'^2_{(1)})\right) \tag{4.11}$$

such that the second order contribution to the energy, after some straightforward but tedious algebra, becomes

$$\delta\mathcal{E}_{(2)}=\frac{1}{\alpha'}\left(-\sqrt{\alpha'}\alpha^1_0+\frac{\alpha'}{R_0}\sum_{n=2}^\infty n(a^{2*}_n a^2_n+b^{2*}_n b^2_n)\right.$$

$$+\frac{\alpha'}{R_0}\sum_{n=2}^\infty\left(\frac{2n^4-2n^2+1}{2n(n^2-1)}(a^{1*}_n a^1_n+b^{1*}_n b^1_n)\right.$$

$$\left.\left.+\frac{2n^2-1}{2n(n^2-1)}(a^1_n b^1_n+a^{1*}_n b^{1*}_n)\right)\right). \tag{4.12}$$

The second order zero mode α^1_0 can be set to zero for the same reason as in Eq. (4.9). Then we get the following result for the mass formula up to second order perturbations:

$$M^2\alpha'=\frac{R_0^2}{\alpha'}+2\sum_{n=2}^\infty n(a^{2*}_n a^2_n+b^{2*}_n b^2_n)$$

$$+2\sum_{n=2}^\infty\left(\frac{2n^4-2n^2+1}{2n(n^2-1)}(a^{1*}_n a^1_n+b^{1*}_n b^1_n)\right.$$

$$\left.+\frac{2n^2-1}{2n(n^2-1)}(a^1_n b^1_n+a^{1*}_n b^{1*}_n)\right). \tag{4.13}$$

This is the mass formula for an observer using stringtime τ, i.e., for an observer for which a^i_n and b^i_n are the positive frequency modes. For large n (high frequency), the result reduces to the standard Minkowski result. This was to be expected since the high frequency modes do not ''feel'' that they are living on a contracting macroscopic circular string. For small n (low frequency), the result however differs significantly from the standard Minkowski result. Notice in particular that there are off-diagonal terms in the mass formula.

It is convenient to introduce an alternative set of modes which diagonalizes the mass formula. We take $\tilde{a}^2_n=a^2_n$ and $\tilde{b}^2_n=b^2_n$ but

$$\tilde{a}_n^1 = \frac{1}{2}\sqrt{\frac{n^2}{2(n^2-1)}}\left[\frac{2n^2-1}{n^2}(a_n^1+b_n^1) + \frac{1}{n^2}(a_n^{1*}+b_n^{1*})\right],$$

(4.14)

$$\tilde{b}_n^1 = \frac{-i}{2}\sqrt{\frac{n^2}{2(n^2-1)}}\left[\frac{2n^2-1}{n^2}(a_n^1-b_n^1)\right.$$
$$\left. -\frac{1}{n^2}(a_n^{1*}-b_n^{1*})\right].$$

(4.15)

Then Eq. (4.13) reduces to

$$M^2\alpha' = \frac{R_0^2}{\alpha'} + 2\sum_{n=2}^{\infty} n(\tilde{a}_n^{i*}\tilde{a}_n^i + \tilde{b}_n^{i*}\tilde{b}_n^i), \qquad (4.16)$$

which is the standard Minkowski result, except for the ''zero-point energy'' R_0^2/α' which is due to the unperturbed circular string.

The modes (4.14),(4.15) satisfy the same algebra as in Eq. (3.12), but it is important to stress that they are *not* positive frequency modes for an observer using string time τ. At the quantum level, it means that the vacuum defined in terms of the annihilation operators $\tilde{a}_n^i$, $\tilde{b}_n^i$

$$\tilde{a}_n^i|O\rangle = \tilde{b}_n^i|O\rangle = 0 \qquad (4.17)$$

does not coincide with the vacuum defined in terms of a_n^i, b_n^i. But this is of course a well-known problem when dealing with quantization of fields in a curved spacetime (see for instance [9]); in our case the fields are represented by the pertubations Φ^i while the curved spacetime is represented by the unperturbed contracting circular string.

V. DISCUSSION

We considered the motion of a macroscopic string in flat Minkowski spacetime. A string system is endowed with conformal symmetry on the world sheet and the string equations of motion are supplied with corresponding constraints. The symmetry and the constraints underline the fact that only the transverse string motion is physical. Therefore the actual physical degrees of freedom of a string are $D-2$, rather than D, where D is the dimension of spacetime. There are two alternatives open: Either work with all D fields and check at every stage of the calculation that the constraints are satisfied; or work directly with the transverse $D-2$ fields.

Our case study, a macroscopic string contracting under the influence of its own tension and experiencing small oscillations, is most suited for the second approach, as explained in the Introduction. We adopted the perturbative scheme of Garriga and Vilenkin [2]. An exact special solution is introduced as the zeroth order solution. The perturbation of the zeroth order solution should include only tranverse oscillations and therefore lives in the subspace normal to the unperturbed string world sheet. We improved the perturbative expansion by including second order terms. Since at each successive order the string world sheet is redefined,

we have to redefine also the normal vectors to the world sheet. Our expressions for the second order perturbation are compact and are expressed covariantly in terms of geometric quantities.

Our formalism was exemplified by studying an almost circular string in Minkowski spacetime. To zeroth order we have a circular string in the x-y plane. We analyzed to first and second order the transverse oscillations: radial oscillations and oscillations in the z direction. The z oscillations appear quite standard (the usual oscillations in flat spacetime). On the other hand, the radial oscillations present novel features. The frequency spectrum of the radial oscillations differs from the standard one for small and moderate frequencies. Furthermore, a nondiagonal mixing appears in the mass spectrum. This reflects that the scalar field (which represents the perturbation) in the radial direction is ''feeling'' the underlying two-dimensional contracting geometry. This is the picture emerging for an observer using world-sheet time τ. However, by using an alternative definition of the vacuum, it is possible to diagonalize the Hamiltonian, and the standard string mass spectrum appears for all frequencies.

It is highly interesting to apply our formalism to strings moving in curved spacetime. Along these lines we studied already [10] an oscillating circular string in Schwarzschild background to zeroth and first order. Our motivation was to establish a framework within which to study in a precise manner the string behavior near the black hole horizon, issues first raised by Susskind [8] (see also [11]). We calculated both the radial and angular spreading of the string, as the string approaches the black hole horizon. We found that the radial spreading is suppressed by the Lorentz contraction and the string appears (to an asymptotic observer) as wrapping around the event horizon. We plan to calculate and include the second order terms and thus analyze how the string oscillators spread over the event horizon. Notice that the second order perturbations are necessary for a consistent discussion of the energy. Hopefully we might understand the entropy of the black hole in terms of string degrees of freedom.

APPENDIX

The first and second order perturbations of x^μ are

$$\delta x_{(1)}^\mu = \bar{n}_i^\mu \Phi_{(1)}^i, \qquad (A1)$$

$$\delta x_{(2)}^\mu = \bar{n}_i^\mu \Phi_{(2)}^i - (\bar{D}_{ij}^A \Phi_{(1)}^j)\Phi_{(1)}^i \bar{x}_{,A}^\mu. \qquad (A2)$$

It follows that

$$(\delta x_{(1)}^\mu)_{,B} = (\bar{D}_{jB}^i \Phi_{(1)}^j)\bar{n}_i^\mu - \bar{\Omega}_{jB}^C \Phi_{(1)}^j \bar{x}_{,C}^\mu, \qquad (A3)$$

$$(\delta x_{(2)}^\mu)_{,B} = (\bar{D}_{jB}^i \Phi_{(2)}^j)\bar{n}_i^\mu - \bar{\Omega}_{jB}^C \Phi_{(2)}^j \bar{x}_{,C}^\mu$$
$$- (\bar{D}_{ji}^A \Phi_{(1)}^i)\bar{\Omega}_{AB}^k \Phi_{(1)}^j \bar{n}_k^\mu - (\bar{D}_{ji}^A \Phi_{(1)}^i)$$
$$\times (\bar{D}_{kB}^j \Phi_{(1)}^k)\bar{x}_{,A}^\mu - (\bar{D}_k^{jB}\bar{D}_{ji}^A \Phi_{(1)}^i)\Phi_{(1)}^k \bar{x}_{,A}^\mu.$$
$$(A4)$$

As for the first and second order perturbations of the (inverse) induced metric on the world-sheet, we get

$$\delta G_{(1)}^{AB} = 2\bar{\Omega}_j^{AB}\Phi_{(1)}^j, \tag{A5}$$

$$\delta G_{(2)}^{AB} = 2\bar{\Omega}_j^{AB}\Phi_{(2)}^j + (\bar{D}_{ji}^A \Phi_{(1)}^i)(\bar{D}_{kB}^j \Phi_{(1)}^k)$$
$$+ 3\bar{\Omega}_j^{Ac}\bar{\Omega}_i^{BC}\Phi_{(1)}^j \Phi_{(1)}^i + (\bar{D}_{jk}^A \bar{D}_i^{kB}\Phi_{(1)}^i)\Phi_{(1)}^j$$
$$+ (\bar{D}_{jk}^B \bar{D}_i^{kA}\Phi_{(1)}^i)\Phi_{(1)}^j. \tag{A6}$$

The following expressions for the first and second order perturbations of $\sqrt{-G}$ are also useful:

$$\delta\sqrt{-G}_{(1)} = -\sqrt{-\bar{G}}\,\bar{\Omega}_{jC}^C \Phi_{(1)}^j, \tag{A7}$$

$$\delta\sqrt{-G}_{(2)} = -\sqrt{-\bar{G}}\Bigg(\bar{\Omega}_{jC}^C \Phi_{(2)}^j + (\bar{D}_k^{jC}\bar{D}_{jiC}\Phi_{(1)}^i)\Phi_{(1)}^k$$
$$+ \frac{1}{2}(\bar{D}_{jC}^k \Phi_{(1)}^j)(\bar{D}_{ki}^{\ \ C}\Phi_{(1)}^i)$$
$$- \frac{1}{2}(\bar{\Omega}_{jC}^{\ \ C}\bar{\Omega}_{iD}^{\ \ D} - \bar{\Omega}_{jCD}\bar{\Omega}_i^{\ CD})\Phi_{(1)}^j \Phi_{(1)}^i\Bigg).$$
$$(A8)$$

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